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The heat storage capacity of a periodically heated slab under general boundary conditions

G.E. Cossali

Facoltà di Ingegneria, Università di Bergamo, via Marconi 6, 24044 Dalmine (BG), Italy

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Abstract

The well-known problem of evaluating the dynamic heat storage capacity of a 1D slab is analysed and extended to case of general boundary conditions and general periodic excitation profiles of any form. A general relation to calculate the storage capacity is obtained in closed form for the harmonic case. Equations in time and frequency domain to calculate the storage capacity under periodic (non-harmonic) heating are also obtained. Some particular cases are analysed and examples are given to show the applicability of the analysis.

Keywords: Heat storage; 1D slab; Periodic conduction; Multilayered slab; Thermal quadrupoles

1. Introduction

Heat storage is an important issue in many applied fields and it is of particular interest for the energy saving in passive buildings [1,2]. In recent years a conspicuous amount of experimental and theoretical work was performed by many researchers on this problem (see for example [3–5]) and other related ones [6–9], with the aim of deepening the understanding of this energy storage method. Magyari and Keller [4] have given an exhaustive analysis of the problem of heat storage in a slab harmonically excited to one end and insulated to the other one, showing that an optimum slab thickness exist (for each frequency) that maximises the heat content. Those results were extended by [10] to the case of periodic non-harmonic heating, again for the same boundary conditions and it was shown that the harmonic heating is not the optimum way to store energy in a 1D slab as other heating profiles yield larger heat storage. The theoretical interest of these results is limited by the fact that the studied boundary conditions are not representative of real cases. The main aim of the present work is to develop a general scheme to analyse the problem of heat storage in 1D slabs, thus extending the previous results of [4,10] to general (linear) boundary conditions and general periodic heating profiles. To this end, equations in frequency and time domain are developed for the most general case and used to evaluate the maximum energy storage for some particular conditions.

2. Basic equations

Consider a homogeneous incompressible 1D slab, the energy conservation equation can be written as:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{1}$$

As the aim is to analyse the periodic regime, it is convenient to split the temperature and flux fields into time-average and fluctuating parts and to introduce the Fourier transforms of the latter as:

$$T(t,x) = T_a(x) + \int_{-\infty}^{+\infty} S(x,\omega) e^{i\omega t}\, d\omega$$
$$q(t,x) = q_a(x) + \int_{-\infty}^{+\infty} Q(x,\omega) e^{i\omega t}\, d\omega \tag{2}$$

where T_a, q_a are the time average values, $T_a(x)$ satisfies the time independent Fourier equation and the most general solution is $T_a(x) = C_1 x + C_2$. The Fourier transform $S(x,\omega)$

E-mail address: gianpietro.cossali@unibg.it.

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Nomenclature

B	non-dimensional convective heat transfer coefficient
Bi	Biot number
c	specific heat
C_1, C_2	constants
d	slab thickness
E	heat storage
h	convective heat transfer coefficient
H	fluctuation amplitude
k	thermal conductivity
K	function
q	heat flux
Q	Fourier transform of heat flux
S	Fourier transform of temperature
t	time
T	temperature
$\mathcal{T}$	period
u, U	internal energy
x	coordinate
Z	Fourier transform of external temperature fluctuations

Greek symbols

α	thermal diffusivity: $\frac{k}{\rho c}$
α_r	absorptivity
β	constant
Γ	Fourier transform of irradiation fluctuations
ε	effusivity: $\varepsilon = \sqrt{k\rho c}$
ε_r	emittivity
η	square of non-dimensional slab thickness: $\eta = \xi^2$
λ	non-dimensional frequency
ξ	non-dimensional slab thickness
ρ	density
σ	Stefan–Boltzman constant: $5.67 \times 10^{-8}\ \mathrm{W\,m^{-2}\,K^{-4}}$
τ	non-dimensional time
Ψ, Λ	non-dimensional functions
ω	frequency

Indexes

a	time average
d	slab of finite thickness
h	harmonic
i	intermittent
in	inner
ou	outer
0	slab surface at $x = 0$
∞	semi-infinite wall

satisfies instead the ODE (where apex means derivation respect to x):

$$i\omega S(x,\omega) = \alpha S''(x,\omega)$$

The general solution can be written as:

$$S(x,\omega) = S(0,\omega)\cosh(\beta x) - \frac{Q(0,\omega)}{k\beta}\sinh(\beta x) \tag{3}$$

with $\beta = \sqrt{\frac{i\omega}{\alpha}}$ and $Q(0,\omega) = -kS'(0,\omega)$.

The internal energy stored into the slab at time t can be evaluated, for a finite slab, through the integral:

$$U_d(t) = U_{d,a} + u_d(t) = \rho c\int_0^d T_a(x)\,dx + \rho c\int_0^d T'(x,t)\,dx$$

where $U_{d,a} = \rho c(\frac{C_1 d^2}{2} + C_2 d)$ is the average internal energy content, $T'(x,t)$ is the fluctuating part of the temperature field, and using (2) and (3):

$$u_d(t) = \int_{-\infty}^{+\infty} \Omega_d(\omega)e^{i\omega t}\,d\omega \tag{4}$$

with

$$\Omega_d(\omega) = -\frac{1}{i\omega}\big[Q(d,\omega) - Q(0,\omega)\big] \tag{5}$$

consistently with the obvious observation that:

$$\frac{du_d(t)}{dt} = q'(0,t) - q'(d,t) = \int_{-\infty}^{+\infty}\big[Q(0,\omega) - Q(d,\omega)\big]e^{i\omega t}\,d\omega$$

Finally, the heat storage capacity can be calculated after finding the minimum and maximum values of the instantaneous energy content over a period:

$$E_d = U_{d,\max} - U_{d,\min} = u_{d,\max} - u_{d,\min}$$

3. The general solution

It is convenient to use the thermal quadrupole formalism [11] and write the general solution under the form:

$$\begin{aligned} S(d,\omega) &= S(0,\omega)\cosh(\beta d) - \frac{Q(0,\omega)}{k\beta}\sinh(\beta d) \\ Q(d,\omega) &= -k\beta S(0,\omega)\sinh(\beta d) + Q(0,\omega)\cosh(\beta d) \end{aligned} \tag{6}$$

or more compactly:

$$Y = \mathbf{M}X \tag{7}$$

with

$$X = \begin{bmatrix} S(0) \\ Q(0,\omega) \end{bmatrix}; \qquad Y = \begin{bmatrix} S(d) \\ Q(d,\omega) \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} \cosh(\beta d) & -\frac{1}{k\beta}\sinh(\beta d) \\ -k\beta\sinh(\beta d) & \cosh(\beta d) \end{bmatrix}$$

the linear relation (7) between the vectors X_j and Y_j (with $j = 1, 2$) is the basis of the thermal quadrupole approach. The Fourier transform of the instantaneous energy content fluctuation $u_d(t)$ is then (see Eqs. (4) and (5)):

$$\Omega_d(\omega) = -\frac{1}{i\omega}(Y_2 - X_2)$$

Now, the most general form of the boundary conditions for the 1D slab can be written as:

$$x = 0; \quad F_0(X_1, X_2) = 0$$
$$x = d; \quad F_d(Y_1, Y_2) = 0$$

For boundary conditions of first, second and third kind the functions F_j are linear, non-linearities are introduced when, for example, radiative boundary conditions are considered, however such conditions can be linearised whenever the surface temperature fluctuations are small compared to the time-averaged absolute temperature [13]. The linearised boundary conditions can then be written as:

$$A^T X = \Pi_0(\omega); \qquad B^T Y = \Pi_d(\omega) \tag{8}$$

with

$$A^T = [a_1, a_2]; \qquad B^T = [b_1, b_2]$$

where a_j, b_j Π_o, Π_d are functions of ω. To be noticed that Π_o, Π_d are the Fourier transforms of the forcing variables fluctuation, like for example the fluid temperature for convective conditions, or the heat flux imposed on the surface for b.c. of second kind, etc. Applying Eq. (7) to the second of Eqs. (8) we obtain:

$$C^T X = \Pi_d$$

with

$$\begin{aligned} C^T &= B^T \mathbf{M} \\ &= \Big[b_1 \cosh(\beta d) - b_2 k\beta \sinh(\beta d), \\ &\qquad -\frac{b_1}{k\beta}\sinh(\beta d) + b_2 \cosh(\beta d) \Big] \end{aligned}$$

and the boundary conditions (8) can be written in compact form as:

$$\mathbf{Q} X = \Pi$$

with

$$\Pi = \begin{bmatrix} \Pi_0 \\ \Pi_d \end{bmatrix}; \qquad \mathbf{Q} = \begin{bmatrix} a_1 & a_2 \\ c_1 & c_2 \end{bmatrix}$$

Observing that the inverse of $\mathbf{Q}$ always exists, the vectors X and Y can be linearly related to the vector Π as:

$$X = \mathbf{Q}^{-1}\Pi, \qquad Y = \mathbf{M}\mathbf{Q}^{-1}\Pi$$

Moreover, the function Ω_d that determines the energy content becomes:

$$\begin{aligned} \Omega_d(\omega) &= -\frac{1}{i\omega}(Y_2 - X_2) \\ &= -\frac{1}{i\omega}\big(-k\beta \sinh(\beta d) X_1 + X_2\big(\cosh(\beta d) - 1\big)\big) \\ &= D^T X = D^T \mathbf{Q}^{-1}\Pi \end{aligned}$$

with

$$D^T = -\frac{1}{i\omega}\big[-k\beta \sinh(\beta d), \cosh(\beta d) - 1\big]$$

With simple algebra one obtains:

$$\Omega_d = C_0(\omega)\Pi_0(\omega) + C_d(\omega)\Pi_d(\omega)$$

with

$$C_0 = \frac{1}{i\omega}\frac{[b_1(1 - \cosh\beta d) + b_2 k\beta \sinh(\beta d)]}{[a_2 b_2 k\beta - \frac{a_1 b_1}{k\beta}]\sinh(\beta d) + [a_1 b_2 - a_2 b_1]\cosh(\beta d)}$$

$$C_d = -\frac{1}{i\omega}\frac{[a_2 k\beta \sinh(\beta d) + a_1(\cosh(\beta d) - 1)]}{[a_2 b_2 k\beta - \frac{a_1 b_1}{k\beta}]\sinh(\beta d) + [a_1 b_2 - a_2 b_1]\cosh(\beta d)}$$

and by applying the convolution theorem [12]:

$$\begin{aligned} u_d(t) &= \int_{-\infty}^{+\infty} \Omega(\omega) e^{i\omega t}\, d\omega \\ &= \int_{-\infty}^{+\infty} \big[C_0(\omega)\Pi_0(\omega) + C_d(\omega)\Pi_d(\omega)\big] e^{i\omega t}\, d\omega \qquad (9) \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} K_0(t - s) P_0(s)\, ds \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{+\infty} K_d(t - s) P_d(s)\, ds \qquad (10) \end{aligned}$$

with

$$K_0(t) = \int_{-\infty}^{+\infty} C_0(\omega) e^{i\omega t}\, d\omega; \qquad K_d(t) = \int_{-\infty}^{+\infty} C_d(\omega) e^{i\omega t}\, d\omega$$

$$P_0(t) = \int_{-\infty}^{+\infty} \Pi_0(\omega) e^{i\omega t}\, d\omega; \qquad P_d(t) = \int_{-\infty}^{+\infty} \Pi_d(\omega) e^{i\omega t}\, d\omega$$

Eqs. (9) and (10) are the general solution of the problem in frequency and time domain respectively. Once the instantaneous energy content is known, the heat storage must be calculated by finding the relative minimum and maximum of $u_d(t)$. For the harmonic case (i.e. the case when the functions $P_0(t)$ and $P_d(t)$ are harmonics) the heat storage capacity is easily found as in this case:

$$\Pi_0(\omega) = \widetilde{\Pi}_0 \delta(\omega - \omega_1); \qquad \Pi_d(\omega) = \widetilde{\Pi}_d \delta(\omega - \omega_1)$$

and the internal energy content becomes:

$$u_d(t) = \mathrm{Re}\big\{\big[C_0(\omega_1)\widetilde{\Pi}_0 + C_d(\omega_1)\widetilde{\Pi}_d\big] e^{i\omega_1 t}\big\}$$

i.e. the instantaneous energy content fluctuation is harmonic with amplitude

$$H = \big|C_0(\omega_1)\widetilde{\Pi}_0 + C_d(\omega_1)\widetilde{\Pi}_d\big|$$

and the heat storage is simply:

$$\begin{aligned} E_d &= u_{d,\max} - u_{d,\min} \\ &= 2H = 2\big|C_0(\omega_1)\widetilde{\Pi}_0 + C_d(\omega_1)\widetilde{\Pi}_d\big| \end{aligned} \tag{11}$$

4. Some special cases

The above reported analysis yields the general solution for the evaluation of the heat storage capacity of a 1D homogeneous slab under periodic heating. In this section some particular interesting cases will be analysed and Eqs. (9), (10) will be used to derive the particular solutions.

Consider the simplest case of a semi-infinite slab, then:

$$Y_2 = 0; \quad \text{for } d \to \infty$$

the boundary conditions at infinity impose

$$b_1 = 0; \qquad b_2 = 1; \qquad \Pi_d = 0$$

and

$$C_0 = \frac{1}{i\omega} \frac{(\frac{1}{\cosh(\beta d)} - 1)}{[-\frac{a_1}{k\beta}]\tanh(\beta d) + [-a_2]} \to \frac{\rho c}{\beta} \frac{1}{a_1 + a_2 k\beta}$$

$$u_\infty(t) = \rho c \int_{-\infty}^{+\infty} \frac{1}{a_2 k\beta + a_1} \frac{1}{\beta} \Pi_0(\omega) e^{i\omega t}\, d\omega$$

that gives the most general result for the semi-infinite slab. For the case of harmonic heating (see Eq. (11)):

$$\begin{aligned} E_d &= 2\big|C_0(\omega_1)\big|\big|\widetilde{\Pi}_0\big| \\ &= 2\frac{\varepsilon}{\sqrt{\omega_1}} \frac{1}{\sqrt{a_1^2 + \sqrt{2}a_1 a_2 \varepsilon\sqrt{\omega_1} + a_2^2\varepsilon^2\omega_1}} |\widetilde{\Pi}_0| \end{aligned} \tag{12}$$

generalising the well-known solution for the simplest case of boundary conditions of first kind: $E_d = \frac{2\varepsilon}{\sqrt{\omega}} T_0$ (with $\varepsilon = \sqrt{\rho c k}$ and T_0 the amplitude of temperature fluctuation, see also [4]) that stems from (12) when: $a_1 = 1$, $a_2 = 0$. The interesting case of convecting heating is instead obtained by setting:

$$a_1 = 1; \qquad a_2 = \frac{1}{h}$$

where h is the convective coefficient and $T_f(t) = T_{f,a} + \int_{-\infty}^{+\infty} Z_0(\omega) e^{i\omega t}\, d\omega$ is the fluid temperature, then $\Pi_0 = Z_0$ and defining $\lambda = \frac{\omega}{2}\frac{\varepsilon^2}{h^2}$

$$\begin{aligned} C_0 &= \frac{\rho c}{\beta} \frac{1}{1 + \frac{k\beta}{h}} = \frac{k\rho c}{h\frac{k\beta}{h}} \frac{1}{1 + \frac{k\beta}{h}} \\ &= \frac{k\rho c}{h} \frac{1}{\sqrt{2i}\sqrt{\lambda}(1 + \sqrt{2i}\sqrt{\lambda})} \end{aligned} \tag{13}$$

As observed in [10] the heating profiles play an important role in defining the maximum heat storage and it was shown that intermittent heating may increase the heat storage of a factor of about 1.52 respect to harmonic heating, when boundary conditions of first kind are considered. Similar results can now be obtained for the more general case of convecting heating. The fluid temperature may be considered to vary sinusoidally or in an intermittent way with the same fluctuating amplitude and period $\mathcal{T}$, in analytic form:

$$T'_{f,i}(t) = \begin{cases} T_0 & \text{for } 0 < t < t_i \\ -T_0 & \text{for } t_i < t < \mathcal{T} \end{cases} \quad \text{intermittent}$$

$$T'_{f,h}(t) = T_0 \sin\left(\frac{2\pi}{\mathcal{T}} t\right) \quad \text{harmonic} \tag{14}$$

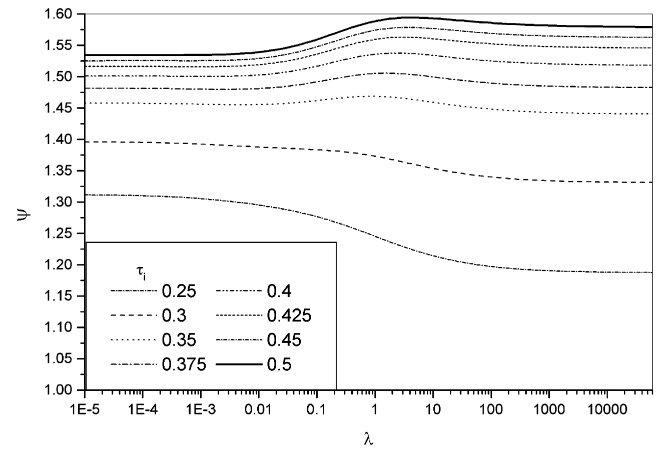

Fig. 1. Ratio between heat storage in a semi-infinite slab under intermittent convective condition and under harmonic conditions, as a function of non-dimensional frequency.

The numerical evaluation of the inverse Fourier transform (by a standard FFT algorithm) allows to evaluate the energy content and then the heating storage for the intermittent case, whereas for the harmonic heating instead Eq. (11) gives:

$$E_d = \frac{k\rho c}{h} \left|\frac{\sqrt{2}}{\sqrt{\lambda}}\right| \left|\frac{1}{1 + 2\sqrt{\lambda} + 2\lambda}\right|^{1/2} T_0$$

The ratio $\Psi = \frac{E_i}{E_h}$ between the heat storage under intermittent and harmonic heating depends on the frequency and on h through the parameter λ. Fig. 1 shows the results obtained for different values of the parameter $\tau_i = \frac{t_i}{\mathcal{T}}$. It is worth to notice that the limit $h \to \infty$ ($\lambda \to 0$) represents the case of boundary condition of first kind (imposed surface temperature), i.e. the problem treated by [4,10].

Consider now the case of a finite thickness slab subject to periodic convective heating on $x = 0$ (with fluid temperature fluctuation) and steady convection on $x = d$ (with fixed fluid temperature). The boundary conditions for the periodic problem are:

$$h_0\big(Z_0(\omega) - S(0,\omega)\big) = Q(0,\omega)$$
$$h_d S(d,\omega) = Q(d,\omega)$$

and

$$A^T = \left[1, \frac{1}{h_0}\right]; \qquad B^T = \left[1, -\frac{1}{h_d}\right]; \qquad \Pi = \big[Z_0(\omega), 0\big]$$

where $T_f(t) = T_{f,a} + \int_{-\infty}^{+\infty} Z_0(\omega) e^{i\omega t}\, d\omega$ is the imposed fluid temperature. Then the energy content for the harmonic heating case ($Z(\omega) = T_f \delta(\omega - \omega_1)$) can be evaluated through Eq. (11):

$$E_d = 2\left|\frac{\rho c}{\beta} \frac{[\sinh(\sqrt{2i}\xi) - \frac{B_d}{\sqrt{i}}(1 - \cosh(\sqrt{2i}\xi))]}{[\frac{\sqrt{i}}{B_0} + \frac{B_d}{\sqrt{i}}]\sinh(\sqrt{2i}\xi) + [1 + \frac{B_d}{B_0}]\cosh(\sqrt{2i}\xi)}\right| T_f$$

with $B_0 = \frac{h_0}{\varepsilon\sqrt{\omega}}$; $B_d = \frac{h_d}{\varepsilon\sqrt{\omega}}$, and $\xi = \sqrt{\frac{\omega}{2\alpha}}d$. It should be pointed out that after defining the "penetration depth" (see also [11]) as $d_p = \sqrt{\frac{2\alpha}{\omega}}$ the non-dimensional variable ξ can be seen as the ratio between the actual slab thickness and the penetration depth, while the non-dimensional numbers $B_x = \frac{h_x}{\varepsilon\sqrt{\omega}}$ can

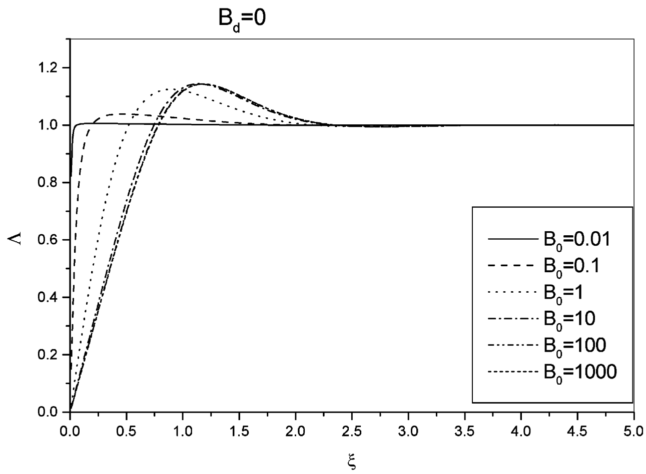

Fig. 2. Ratio between heat storage in finite and semi-infinite slabs under convective condition and adiabatic end, as a function of non-dimensional slab thickness.

be written as $B_x = \frac{h_x d_p}{k}$ and they can be seen as Biot numbers calculated on the penetration depth. Noticing that for a semi-infinite slab under the same conditions we have:

$$E_\infty = 2\left|\frac{\rho c}{\beta}\frac{1}{\frac{\sqrt{i}}{B_e}+1}\right| T_f$$

then:

$$E_d = \Lambda_d E_\infty$$

with

$$\Lambda_d = \left|\frac{\sqrt{i}}{B_0}+1\right| \times \left|\frac{[\sinh(\sqrt{2i}\xi) - \frac{B_d}{\sqrt{i}}(1-\cosh(\sqrt{2i}\xi))]}{[\frac{\sqrt{i}}{B_0}+\frac{B_d}{\sqrt{i}}]\sinh(\sqrt{2i}\xi) + [1+\frac{B_d}{B_0}]\cosh(\sqrt{2i}\xi)}\right|$$

Fig. 2 shows Λ_d as a function of the nondimensional slab thickness ξ for different values of B_0 and $B_d = 0$ (i.e. adiabatic conditions on $x = d$). The special case $B_d = 0$ and $B_0 = \infty$ (i.e. imposed periodic temperature fluctuation on $x = 0$ and adiabatic conditions on $x = d$) is the case treated by [4] for the harmonic heating and by [10] for general non-harmonic heating profiles. It is seen that moving from imposed temperature conditions ($B_0 = \infty$) to convective ones the optimum slab thickness (the one yielding the maximum heat storage) diminish as well as the maximum heat storage.

It is of some interest the "dual" problem set by imposing first kind b.c. on $x = 0$ and convective conditions on $x = d$ (with fixed fluid temperature), i.e.:

$$S(0,\omega) = S_0(\omega); \qquad h_d S(d,\omega) = Q(d,\omega)$$

where $T_0(t) = T_{0,a} + \int_{-\infty}^{+\infty} S_0(\omega)e^{i\omega t}\,d\omega$ is the imposed surface temperature at $x = 0$, then:

$$A^T = [1, 0]; \qquad B^T = \left[1, -\frac{1}{h_d}\right]; \qquad \Pi = [S_0, 0]$$

but it is easy to see that this case is simply obtained from the previous one by setting $B_0 = \infty$ and Fig. 3 shows the results.

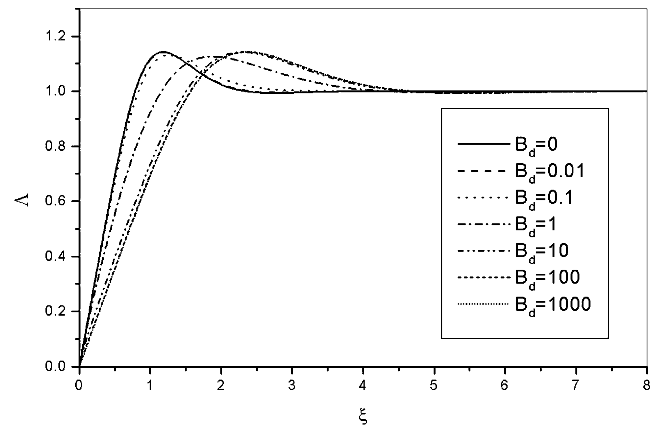

Fig. 3. Ratio between heat storage in finite and semi-infinite slab under imposed fluctuating temperature ($B_0 = \infty$) and convective condition on $x = d$, as a function of non-dimensional slab thickness.

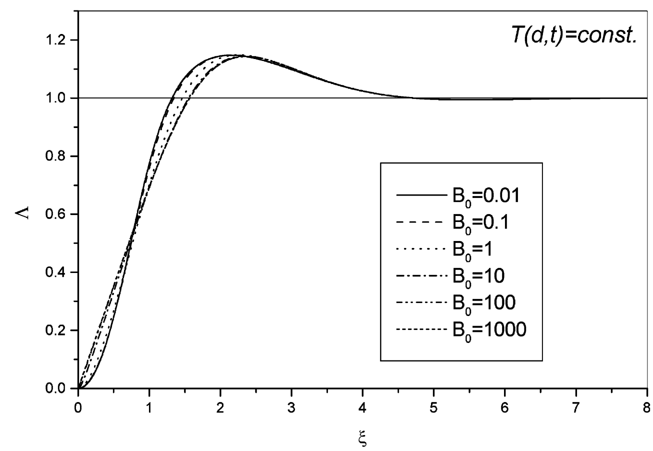

Fig. 4. Ratio between heat storage in finite and semi-infinite slab under convective condition and isothermal end, as a function of non-dimensional slab thickness.

The increase of the convective coefficient at $x = d$ increases the optimum wall thickness but leaves almost unchanged the maximum heat storage.

Finally the case with imposed isothermal condition at $x = d$ and convective conditions on $x = 0$ is reported in Fig. 4, showing that in this case the maximum heat storage does not change significantly with convective coefficient and the optimum slab thickness is only weakly affected.

4.1. Examples of application

To apply the above developed method, we will consider two examples related to the problem of energy storage in buildings. Consider first the case treated in [3] where a horizontal concrete slab ($d = 0.1$ m) was exposed on one side to an intermittent radiative heating (duration of the illumination: 18 hours, irradiation: 381.7 $\mathrm{W\,m^{-2}}$), and kept insulated on the other side. The wall temperature in different points of the slab was measured during both charge and discharge period and the results were reported. The experiment was actually carried on as a typical case of impulsive heating rather than periodic, as the charge and discharge period were not repeated. Despite this conceptual difference between the experiments and the present theory, it is possible to approximate the experiment conditions by a periodic intermittent heating with a discharging period long enough to reach, just before the begin of the charging period, a temperature very close to the initial temperature of the experiment. For this case the boundary conditions are:

$$x=0 \quad \alpha_r G(t) - \sigma\varepsilon_r T^4(0,t) + h\big(T_f - T(0,t)\big) = -k\left(\frac{\partial T}{\partial x}\right)_{x=0}$$
$$x=d \quad 0 = -k\left(\frac{\partial T}{\partial x}\right)_{x=d} \tag{15}$$

where (T_f) is the temperature of the air in contact with the illuminated surface and $G(t)$ is the radiative flux. It must be mentioned that the air temperature T_f was varying during the experiment (as reported in [3]) but the values are not given, therefore in the present case we will consider it constant to a standard value of 25 °C. Linearising the first of Eq. (15) under the conditions that the surface temperature fluctuation amplitude (at $x=0$) is small compared to the absolute time-average value $T_a(0) = T_{a,0}$, the boundary conditions for the periodic problem are:

$$\alpha_r \Gamma(\omega) - \varepsilon_r \sigma T_{a,0}^3 S(0,\omega) - hS(0,\omega) = Q(0,\omega)$$
$$0 = Q(d,\omega)$$

where and $G(t) = G_a + \int_{-\infty}^{+\infty} \Gamma(\omega) e^{i\omega t}\, d\omega$. The linearisation of the radiative term led to a relative error in the estimation of the maximum and minimum values of it by about 16%, while the effect on the total heat transfer coefficient $h_T = (\varepsilon_r \sigma T_{a,0}^3 + h)$ is estimated to be lower than 4%. The convective heat transfer coefficient was calculated from the correlation suggested in [3], the Rayleigh number was calculated using the average temperature of 321 K, calculated from the data reported in the cited paper. Using the above discussed formalism the boundary conditions become:

$$X_1 + \frac{1}{h_T} X_2 = \frac{\alpha_r \Gamma(\omega)}{h_T}; \qquad Y_2 = 0$$

yielding:

$$a_1 = 1; \qquad a_2 = \frac{1}{h_T}; \qquad \Pi_0 = \frac{\alpha_r \Gamma(\omega)}{h_T}$$
$$b_1 = 0; \qquad b_2 = 1; \qquad \Pi_d = 0$$

and

$$C_0 = \frac{\rho c d}{\sqrt{2i\eta}} \frac{\tanh(\sqrt{2i}\sqrt{\eta})}{\{Bi^{-1}\sqrt{2i\eta}\tanh(\sqrt{2i}\sqrt{\eta}) + 1\}} = \rho c d C_{0,ad}$$

where $\eta = \frac{\omega d^2}{2\alpha}$, $Bi = \frac{d h_T}{k}$. Finally the energy content fluctuation is:

$$u_d(t) = \rho c d \int_{-\infty}^{+\infty} C_{0,ad}(\omega) \Pi_0(\omega)\, d\omega$$

To notice that the surface temperature in $x=0$ and $x=d$ can be obtained as:

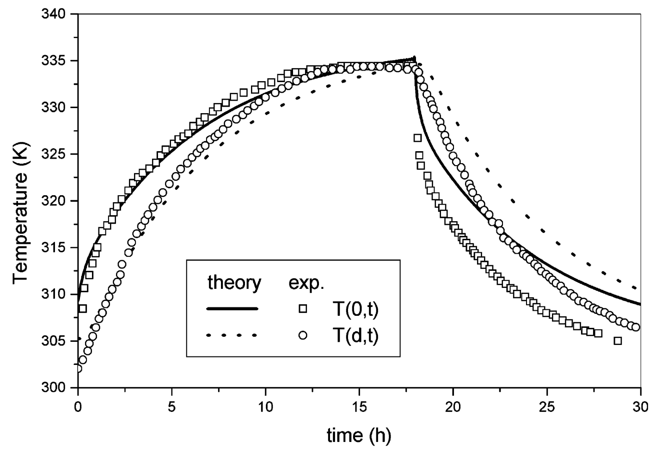

Fig. 5. Comparison of the theoretical calculation of surface temperature of a concrete slab and the experimental results reported by [3].

$$T(0,t) = T_a(0) + \int_{-\infty}^{+\infty} X_1(\omega) e^{i\omega t}\, d\omega$$
$$T(d,t) = T_a(d) + \int_{-\infty}^{+\infty} Y_1(\omega) e^{i\omega t}\, d\omega$$

with

$$X_1 = \frac{1}{B_0^{-1}\sqrt{2i}\sqrt{\eta}\tanh(\sqrt{2i}\sqrt{\eta}) + 1} \Pi_0$$
$$Y_1 = \frac{1}{\cosh(\sqrt{2i}\sqrt{\eta})} \left\{ \frac{1}{B_0^{-1}\sqrt{2i}\sqrt{\eta}\tanh(\sqrt{2i}\sqrt{\eta}) + 1} \right\} \Pi_0$$

and Fig. 5 reports the results compared to those reported in [3] (obtained from the figure reported in the paper). The qualitative agreement is quite good considering the approximations introduced and the above mentioned missing of information.

As a second example consider the case of a wall dividing an inner fluid, kept at a fixed temperature (T_{in}), from an outer fluid with varying temperature ($T_{\text{ou}}(t)$), and suppose that the outer wall surface is heated by radiation. This may represent a typical problem of energy storage in buildings. The boundary conditions are now:

$$x=0 \quad \alpha_r G(t) - \sigma\varepsilon_r T^4(0,t) + h_{\text{ou}}\big(T_{\text{ou}}(t) - T(0,t)\big) = -k\left(\frac{\partial T}{\partial x}\right)_{x=0}$$
$$x=d \quad h_{\text{in}}\big(T(d,t) - T_{\text{in}}(t)\big) = -k\left(\frac{\partial T}{\partial x}\right)_{x=d} \tag{16}$$

and again, after linearising the first of Eq. (16) the boundary conditions for the periodic problem are:

$$\alpha_r \Gamma(\omega) - \varepsilon_r \sigma T_{a,0}^3 S(0,\omega) + h_{\text{ou}}\big(Z(\omega) - S(0,\omega)\big) = Q(0,\omega)$$
$$h_{\text{in}} S(d,\omega) = Q(d,\omega)$$

where $T_{\text{ou}}(t) = T_{\text{ou},a} + \int_{-\infty}^{+\infty} Z(\omega) e^{i\omega t}\, d\omega$ and $T_{a,0} = T_a(0)$. Using the above discussed formalism:

$$X_1 + \frac{1}{h_{\text{ou},T}} X_2 = \frac{h_{\text{ou}} Z(\omega) + \alpha_r \Gamma(\omega)}{h_{\text{ou},T}}$$

$$Y_1 - \frac{1}{h_{\text{in}}} Y_2 = 0$$

with $h_{\text{ou},T} = \varepsilon_r \sigma T_{a,0}^3 + h_{\text{ou}}$, then

$$A^T = \left\{1, \frac{1}{h_{\text{ou},T}}\right\}; \qquad B^T = \left\{1, -\frac{1}{h_{\text{in}}}\right\}$$

$$\Pi = \begin{bmatrix} \frac{h_{\text{ou}} Z(\omega) + \alpha_r \Gamma(\omega)}{h_{\text{ou}.T}} \\ 0 \end{bmatrix}$$

As the solar radiation variation is usually modelled as an harmonic function of time, and considering a similar model also for the external air temperature, we may set $Z(\omega) = Z\delta(\omega - \omega_0)$; $\Gamma(\omega) = \Gamma\delta(\omega - \omega_0)$, to be noticed that the fluctuating variables $G'(t)$ and $T_{\text{ou}}(t)$ need not to be in phase. Then

$$\Pi_0(\omega) = \frac{h_{\text{ou}} Z + \alpha_r \Gamma}{h_{\text{ou},T}} \delta(\omega - \omega_0) = \widetilde{\Pi}_0 \delta(\omega - \omega_0)$$

and the problem solution falls inside those presented in the previous section after defining: $B_0 = \frac{h_{\text{ou}}}{\varepsilon\sqrt{\omega}}$; $B_d = \frac{h_{\text{in}}}{\varepsilon\sqrt{\omega}}$.

5. The multilayered slab

The realistic case of multilayered slab can be treated with a similar formalism. The first observation is that Eq. (7) holds for each slab: $Y^{(n)} = \mathbf{M}^{(n)} X^{(n)}$, then the interface conditions allow to link the solutions:

$$X^{(j+1)} = Y^{(j)}$$

For a n-layered slab:

$$\begin{aligned} Y^{(n)} &= \mathbf{M}^{(n)} X^{(n)} = \mathbf{M}^{(n)} Y^{(n-1)} \\ &= \mathbf{M}^{(n)} \mathbf{M}^{(n-1)} Y^{(n-2)} = \widetilde{\mathbf{M}} X^{(1)} \end{aligned}$$

where $\widetilde{\mathbf{M}} = \mathbf{M}^{(n)} \mathbf{M}^{(n-1)} \cdots \mathbf{M}^{(1)}$, the instantaneous energy content of each slab is:

$$u_j(t) = \int_{-\infty}^{+\infty} \Omega_j(\omega) e^{i\omega t}\, d\omega = -\int_{-\infty}^{+\infty} \frac{1}{i\omega} \left[Y_2^{(j)} - X_2^{(j)}\right] e^{i\omega t}\, d\omega$$

the total energy content is:

$$\begin{aligned} u_T(t) &= \sum_{j=1}^{n} u_j(t) = \int_{-\infty}^{+\infty} \sum_{j=1}^{n} \Omega_j(\omega) e^{i\omega t}\, d\omega \\ &= \int_{-\infty}^{+\infty} \Omega_T(\omega) e^{i\omega t}\, d\omega \end{aligned}$$

with

$$\begin{aligned} \Omega_T(\omega) &= \sum_{j=1}^{n} \Omega_j(\omega) = -\frac{1}{i\omega} \sum_{j=1}^{n} \left[Y_2^{(j)} - X_2^{(j)}\right] \\ &= -\frac{1}{i\omega} \left[Y_2^{(n)} - X_2^{(1)}\right] \end{aligned}$$

The boundary conditions can again be written as:

$$A^T X^{(1)} = \Pi_0; \qquad B^T Y^{(n)} = \Pi_n \tag{17}$$

with

$$A^T = [a_1, a_2]; \qquad B^T = [b_1, b_2] \tag{18}$$

and following the same steps of the previous section a similar result is reached:

$$\begin{aligned} \Omega(\omega) &= C_0 \Pi_0 + C_n \Pi_n \\ C_0 &= -\frac{1}{i\omega} \frac{[b_1(m_{1,1} - 1) + b_2 m_{2,1}]}{a_1(b_1 m_{1,2} + b_2 m_{2,2}) - a_2(b_1 m_{1,1} + b_2 m_{2,1})} \\ C_n &= -\frac{1}{i\omega} \frac{a_1(m_{2,2} - 1) - a_2 m_{2,1}}{a_1(b_1 m_{1,2} + b_2 m_{2,2}) - a_2(b_1 m_{1,1} + b_2 m_{2,1})} \end{aligned}$$

where m_{jk} are the elements of $\widetilde{\mathbf{M}}$ (to notice that $\det\widetilde{\mathbf{M}} = 1$). The problem is then solvable with the same formalism used for a single layer slab.

As an example, consider the following case: a two-layered slab ($n = 2$) subject to harmonic heating (with frequency $f' = \frac{\omega'}{2\pi}$) on one side and adiabatic on the other one. The matrix $\widetilde{\mathbf{M}}$ is then:

$$\begin{aligned} \widetilde{\mathbf{M}} &= \mathbf{M}^{(2)} \mathbf{M}^{(1)} \\ &= \begin{bmatrix} \cosh(\zeta_2)\cosh(\zeta_1) + \frac{\varepsilon_1}{\varepsilon_2}\sinh(\zeta_2)\sinh(\zeta_1) & -\frac{1}{k_1\beta_1}\sinh(\zeta_1)\cosh(\zeta_2) - \frac{1}{k_2\beta_2}\sinh(\zeta_2)\cosh(\zeta_1) \\ -k_2\beta_2\sinh(\zeta_2)\cosh(\zeta_1) - k_1\beta_1\cosh(\zeta_2)\sinh(\zeta_1) & \cosh(\zeta_2)\cosh(\zeta_1) + \frac{\varepsilon_2}{\varepsilon_1}\sinh(\zeta_2)\sinh(\zeta_1) \end{bmatrix} \end{aligned}$$

with $\zeta_j = \sqrt{\frac{i\omega' d_j^2}{\alpha_j}} = \beta_j d_j$, $\beta_j = \sqrt{\frac{i\omega'}{\alpha_j}}$, and the boundary conditions are:

$$\begin{aligned} S(0, \omega) &= T_0 \delta(\omega - \omega') \\ Q(d_1 + d_2, \omega) &= 0 \end{aligned}$$

where T_0 is the amplitude of the temperature fluctuations imposed on the external surface of the first layer. These conditions imply (see Eqs. (17), (18)):

$$\begin{aligned} a_1 &= 1; \qquad a_2 = 0; \qquad \widetilde{\Pi}_0 = T_0 \\ b_1 &= 0; \qquad b_2 = 1; \qquad \widetilde{\Pi}_2 = 0 \end{aligned}$$

and the heat storage is calculated as:

$$\begin{aligned} E &= 2\left|C_0(\omega')\widetilde{\Pi}_0 + C_2(\omega')\widetilde{\Pi}_2\right| = 2\left|\frac{1}{\omega}\frac{m_{2,1}}{m_{2,2}}\right| T_0 \\ &= 2\frac{k_1|\beta_1|}{\omega} T_0 \left|\frac{\frac{\varepsilon_2}{\varepsilon_1}\sinh(\zeta_2)\cosh(\zeta_1) + \cosh(\zeta_2)\sinh(\zeta_1)}{\cosh(\zeta_2)\cosh(\zeta_1) + \frac{\varepsilon_2}{\varepsilon_1}\sinh(\zeta_2)\sinh(\zeta_1)}\right| \end{aligned}$$

Considering that the heat storage for a single layer subject to the same conditions is:

$$E_{d,1} = 2\frac{k_1|\beta_1|}{\omega} T_0 \left|\tanh(\zeta_1)\right|$$

the previous equation can be written as:

$$E = E_{d,1} F$$

where

$$F = \left|\frac{\frac{\varepsilon_1}{\varepsilon_2}\cosh(\zeta_2) + \frac{\sinh(\zeta_2)}{\tanh(\zeta_1)}}{\frac{\varepsilon_1}{\varepsilon_2}\cosh(\zeta_2) + \tanh(\zeta_1)\sinh(\zeta_2)}\right|$$

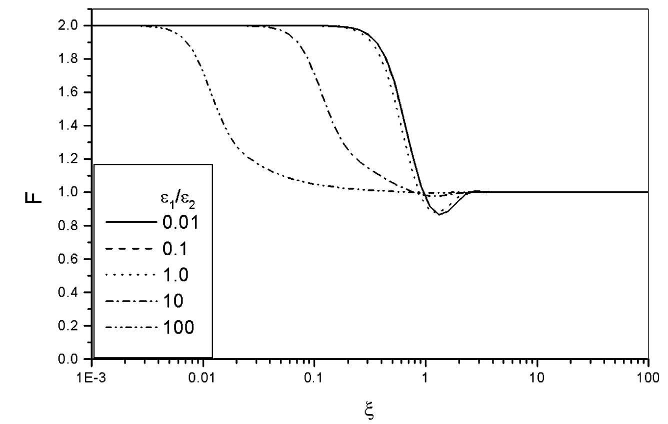

Fig. 6. Two-layered slab heat storage enhancing factor for different values of the effusivity ratio.

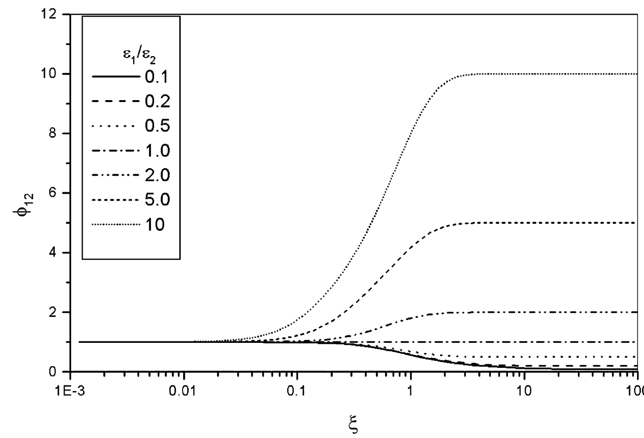

Fig. 7. Heat storage ratio ϕ_{12} for different values of the effusivity ratio.

is the storage enhancing factor due to the addition of the second layer. To understand the physical meaning of F consider the case of two layers of the same heat capacity per square meter ($\rho_1 c_1 d_1 = \rho_2 c_2 d_2$) but different conductivity. Defining $\hat{\varepsilon} = \frac{\varepsilon_1}{\varepsilon_2}$ and $\zeta = \sqrt{2i}\xi = (1+i)\frac{d_1}{d_{p1}}$ where $d_{p_1} = \sqrt{\frac{2\alpha_1}{\omega'}}$ is the penetration depth of the first layer, the storage enhancing factor becomes:

$$F = \left| \frac{\hat{\varepsilon} + \frac{\tanh(\zeta\hat{\varepsilon})}{\tanh(\zeta)}}{\hat{\varepsilon} + \tanh(\zeta)\tanh(\zeta\hat{\varepsilon})} \right|$$

and it is easy to see that for $\omega' \to 0$ (i.e. $\xi \to 0$) the factor $F \to 2$ as in such case the obvious effect of adding a layer of the same capacity it is just the doubling of the wall heat storage, as the wall conductivities become irrelevant. For larger heating frequency instead (i.e. $\xi \to \infty$) the factor $F \to 1$, as in fact under such heating conditions the first layer thickness become much larger that its penetration depth and the thermal response of the wall resemble that of a semi-infinite wall (with the characteristics of the first layer) while the effect of the second one disappears. This example points out the following interesting peculiarity: (i) for low frequency oscillations, such that the penetration depth of both layers is larger that their thickness, the effect of the material conductivity on heat storage is neglectful; (ii) for large frequency oscillations, such that the penetration depth of the first layer is much smaller than its thickness, the heat storage is not influenced by the presence of the second layer; (iii) at intermediate frequencies the effect on heat storage strongly depends on the relative value of the layer conductivities as reported in Fig. 6 for different values of the parameter $\hat{\varepsilon}$. It is also interesting to consider the case when the two layers are switched and to compare the results in terms of heat storage. Define $E_{a,b}$ as the heat storage of the two-layered slab when the first layer is labelled by "a" and the second by "b", then consider the ratio:

$$\phi_{12} = \frac{E_{a,b}}{E_{b,a}}$$

of the heat storages when the layers are switched. From the previous results the ratio can be calculated:

$$\phi_{12} = \frac{E_{ab}}{E_{ba}}$$

$$= \frac{\varepsilon_a \tanh(\zeta_a)}{\varepsilon_b \tanh(\zeta_a \frac{\varepsilon_a}{\varepsilon_b})} \left| \frac{\frac{\varepsilon_a}{\varepsilon_b} + \frac{\tanh(\zeta_a \frac{\varepsilon_a}{\varepsilon_b})}{\tanh(\zeta_a)}}{\frac{\varepsilon_b}{\varepsilon_a} + \frac{\tanh(\zeta_a)}{\tanh(\zeta_a \frac{\varepsilon_a}{\varepsilon_b})}} \right|$$

$$\times \left| \frac{\frac{\varepsilon_b}{\varepsilon_a} + \tanh(\zeta_a \frac{\varepsilon_a}{\varepsilon_b})\tanh(\zeta_a)}{\frac{\varepsilon_a}{\varepsilon_b} + \tanh(\zeta_a)\tanh(\zeta_a \frac{\varepsilon_a}{\varepsilon_b})} \right|$$

and the following limiting conditions hold:

$$\lim_{\xi_a \to 0} \phi_{12} = 1; \qquad \lim_{\xi_a \to \infty} \phi_{12} = \frac{\varepsilon_a}{\varepsilon_b}$$

Moreover, Fig. 7 show that $\phi_{12} > 1$ (for any value of ω) when $\varepsilon_1 > \varepsilon_2$, then the correct arrangement of the two layers (i.e. adiabatic conditions on the lower conductivity material) may increase the heat storage.

6. Conclusions

The general case of periodic (not necessarily harmonic) heating under the most general (linear) boundary conditions for finite and semi-infinite slabs was analysed in terms of heating storage. Equations in time and frequency domain were obtained to calculate the instantaneous energy content and then to calculate the heat storage and applied to some special cases and two concrete example. For a slab of finite thickness with an adiabatic end, the instantaneous energy content was simply related to the analogous content for a semi-infinite slab under the same conditions, for the harmonic heating case an optimum slab thickness, that maximises the heat storage capacity, is shown to exist also for this more general case. Applications to real cases are considered with a comparison with available experimental data. Finally, the use of thermal quadrupole approach allows a straightforward extension of the results to the more general case of multilayer slabs, and an interesting application to a two-layered slab is discussed. As a final remark, attempts to apply the present method to the case of phase change materials, that are widely used to enhance heat storage capacity, have not given appreciable results, mainly due to the non-linearity introduced by the existence of a moving boundary between the solid and molten phase.

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